Time optimal swimmers and Brockett integrator

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Abstract

The aim of this work is to compute time optimal controls for micro-swimmers. The action of swimming is seen as a control problem. More precisely, given an initial position and a target position, can the swimmer move to the target by changing its shape. The motion of the swimmer in the fluid results from the fluid-structure interaction. For micro-swimmers, the fluid equations in consideration are the stationnary Stokes equations. The way of swimming can be described by the following steps: 1. the swimmer modifies its shape, 2. this modification creates a velocity field in the fluid, 3. the fluid velocity acts on the swimmer as a force, 4. the fluid force moves the swimmer. Of course things are not so distinct and the swimming is a highly coupled nonlinear control problem. In this note, we present some key results for a *fast* numerical method to compute time optimal controls for axi-symmetric micro-swimmers. This numerical method is based on explicit formulae of time optimal controls for the Brockett integrator which is a system approaching the dynamic of the swimmer.

AMS subject classification: 34H05, 49J15, 49K15, 49K20, 93C15
Key words: Time optimal controllability, State constraints, Numerical resolution, Brockett integrator, Micro-swimmers

1 Introduction

Understanding the motion of micro-organisms is a challenging issue since at their size the fluid forces are only viscous forces and micro-organisms live in a world where inertia does not exist. Despite the pioneer works modeling and analysing the motions of micro-swimmers (see for instance [26, 16, 15, 22, 9, 25, 24]), the swimming of micro-organisms has only be recently tackled as a control problem. A lot of controllability results for various swimmers has been obtained (see for instance [4, 5, 21, 23, 20] for axi-symmetric swimmers, [3, 18] for general swimmers or [6, 10] when the fluid domain is not the whole space $\mathbb{R}^3$).

In this note, we will consider axi-symmetric micro-swimmers performing small shape changes. For those swimmers, we will study the time optimal controllability of the dynamical system associated to the swimming problem. In particular, this article links the work [20] to [19].

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2 Modeling and problem formulation

We consider a micro-swimmer performing axi-symmetric deformations. Let us denote $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ the canonical basis of $\mathbb{R}^3$ and assume that $\mathbf{e}_1$ represents the symmetry axis. At any time, the swimmer will be diffeomorphic to the unit sphere of $\mathbb{R}^3$ and its shape at rest is the unit sphere S_0.
The control problem is the following: starting from the initial location 0, reach the final position $h^f \mathbf{e}_1$ by shape changes such that at the initial and final positions, the swimmer is the unit sphere of $\mathbb{R}^3$.

2.1 Axi-symmetric coordinates

Since we are considering axi-symmetric swimmers, we introduce the spherical coordinate system $(r, \theta, \phi) \in \mathbb{R}_+ \times [0, \pi] \times [0, 2\pi)$ which will be used all along this note. For every $x = (x_1\ x_2\ x_3)^\top \in \mathbb{R}^3$, the spherical coordinates $(r, \theta, \phi) = \big(r(x), \theta(x), \phi(x)\big)$ are such that:

$$x = r \begin{pmatrix} \cos\theta \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \end{pmatrix} \qquad (x \in \mathbb{R}^3\,, (r, \theta, \phi) \in \mathbb{R}_+ \times [0, \pi] \times [0, 2\pi))\,,$$

with the associated local system of unit vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ given by (see Figure 1):

$$\mathbf{e}_r = \begin{pmatrix} \cos\theta \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \end{pmatrix}\,, \qquad \mathbf{e}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta\cos\phi \\ \cos\theta\sin\phi \end{pmatrix} \quad \text{and} \quad \mathbf{e}_\phi = \begin{pmatrix} \cos\theta \\ -\sin\theta\sin\phi \\ \sin\theta\cos\phi \end{pmatrix}\,.$$

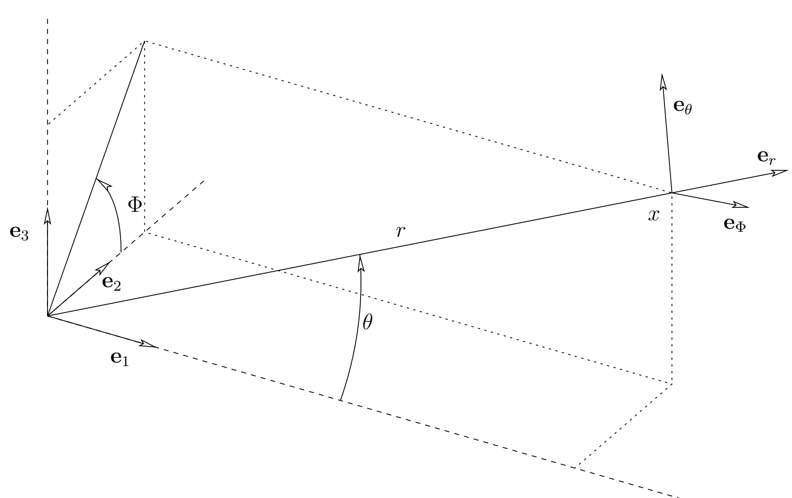


Figure 1: Spherical coordinates.

2.2 Swimmer's deformations

The shape of the swimmer at rest is the unit ball of $\mathbb{R}^3$ which forms the reference shape denoted by S_0. We assume that the deformation of the swimmer is axi-symmetric with respect to the symmetry axis $\mathbf{e}_1$.

More precisely, we assume that the deformation X is built from two elementary deformations D_1 and D_2, that is

$$X(t,x) = x + \alpha_1(t)D_1(x) + \alpha_2(t)D_2(x)\,, \qquad (t \geqslant 0\,,\ x \in \mathbb{R}^3)\,, \tag{2.1}$$

where D_1 and D_2 are axi-symmetric and radial deformations, i.e.

$$D_i(x) = r(x)\delta_i(\theta(x))\mathbf{e}_r(x) \qquad (t \geqslant 0\,,\ x \in \mathbb{R}^3\,,\ i \in \{1,2\})\,, \tag{2.2}$$

with $\delta_i \in C^1([0,\pi],\mathbb{R})$ and $\alpha_i \in L^\infty(\mathbb{R}_+,\mathbb{R})$ such that:

$$\alpha_1(t)\delta_1(\theta) + \alpha_2(t)\delta_2(\theta) > -1 \qquad (t \geqslant 0\,,\ \theta \in [0,\pi])\,, \tag{2.3}$$

so that $X(t,\cdot)$ is a C^1-diffeomorphism on S_0. We define the domain $S(t)$ occupied by the deformed swimmer at time t in the reference frame attached to the swimmer:

$$\mathcal{S}(t) = X(t,S_0) \subset \mathbb{R}^3 \qquad (t \geqslant 0)\,.$$

We also assume that the deformation X does not produce any translation. To this end, we introduce the mass density $\rho_0(x) = 1$ in the shape of the swimmer S_0 at rest and we assume that the mass is locally preserved during the deformation, that is to say that the density of the swimmer at any time $t \geqslant 0$ is given by:

$$\rho(t,X(t,x)) = \frac{1}{|\mathrm{Jac}X(t,x)|} \qquad (t \geqslant 0\,,\ x \in S_0)\,,$$

where $\mathrm{Jac}X(t,\cdot)$ denotes the Jacobian of the mapping $X(t,\cdot)$. According to (2.1), (2.2) we have:

$$\rho(t,X(t,x)) = \frac{1}{(1+\alpha_1(t)\delta_1(\theta(x)) + \alpha_2(t)\delta_2(\theta(x)))^3} \qquad (t \geqslant 0\,,\ x \in S_0)\,.$$

With this mass density, we have, for all $t \geqslant 0$:

$$\int_{\mathcal{S}(t)} \rho(t,x)\,\mathrm{d}x = \int_{S_0} \mathrm{d}x := m_0 \tag{2.4}$$

and the mass center of the swimmer is given by:

$$\begin{aligned}
\int_{\mathcal{S}(t)} x\rho(t,x)\,\mathrm{d}x &= \int_{S_0} (x + \alpha_1(t)D_1(x) + \alpha_2(t)D_2(x))\,\mathrm{d}x \\
&= \int_0^\pi \int_0^1 \int_0^{2\pi} r\,(1+\alpha_1(t)\delta_1(\theta)+\alpha_2(t)\delta_2(\theta))\,\mathbf{e}_r(\theta,\phi)\,r^2\sin\theta\mathrm{d}\phi\mathrm{d}r\mathrm{d}\theta \\
&= \left(\alpha_1(t)\frac{\pi}{2}\int_0^\pi \delta_1(\theta)\cos\theta\sin\theta\,\mathrm{d}\theta + \alpha_2(t)\frac{\pi}{2}\int_0^\pi \delta_2(\theta)\cos\theta\sin\theta\,\mathrm{d}\theta\right)\mathbf{e}_1\,,
\end{aligned}$$

Consequently, we assume:

$$\int_0^\pi \delta_i(\theta)\sin(2\theta)\,\mathrm{d}\theta = 0 \qquad (i \in \{1,2\})\,, \tag{2.5}$$

so that the mass center of the swimmer does not move with the deformation X.

Finally, let us consider the domain $\mathcal{S}^\dagger(t)$ occupied by the swimmer in the fluid at time $t \geqslant 0$, which is given by

$$\mathcal{S}^\dagger(t) = \mathcal{S}(t) + h(t)\mathbf{e}_1. \tag{2.6}$$

Since we have assumed that the deformation X does not introduce any translation, $h(t)\mathbf{e}_1$ is the mass center position of the swimmer in the fluid domain at time t. The domains S_0, $\mathcal{S}(t)$ and $\mathcal{S}^\dagger(t)$ are depicted on Figure 2.

In term of the control theory, $\alpha = (\alpha_1,\ \alpha_2)^\top$ is the system's input and h is its output we aim to control.

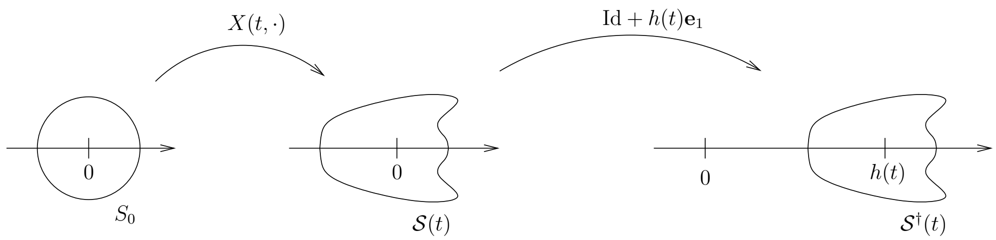


Figure 2: The deformation X and the translation $h\mathbf{e}_1$ of the swimmer.

2.3 Fluid flow

At each time $t \geqslant 0$, the domain occupied by the fluid is $\mathcal{F}^\dagger(t) = \mathbb{R}^3 \setminus \mathcal{S}^\dagger(t)$. The incompressible Navier-Stokes equations reads as:

$$\rho_f \left(\partial_t \boldsymbol{u}^\dagger + \boldsymbol{u}^\dagger \cdot \nabla \boldsymbol{u}^\dagger\right) - \mu\Delta \boldsymbol{u}^\dagger + \nabla p^\dagger = 0 \qquad \text{in } \mathcal{F}^\dagger(t) \tag{2.7a}$$

$$\operatorname{div} \boldsymbol{u}^\dagger = 0 \qquad \text{in } \mathcal{F}^\dagger(t) \tag{2.7b}$$

where $\boldsymbol{u}^\dagger$ is the fluid velocity, $p^\dagger$ is the pressure in the fluid, ρ_f is the mass density and μ is the viscosity of the fluid. We also assume that the fluid is at rest at infinity, i.e.:

$$\lim_{|x|\to\infty} \boldsymbol{u}^\dagger(t,x) = 0\,, \tag{2.7c}$$

and the fluid sticks to the swimmer's boundary, that is to say, for all $x \in \partial\mathcal{S}^\dagger(t)$):

$$\boldsymbol{u}^\dagger(t,x) = \dot{h}(t)\mathbf{e}_1 + \dot{\alpha}_1(t) D_1\left(X(t,\cdot)^{-1}\left(x - h(t)\right)\right) \\ + \dot{\alpha}_2(t) D_2\left(X(t,\cdot)^{-1}\left(x - h(t)\right)\right) \tag{2.8}$$

Let us note $\sigma^\dagger = \mu\left(\nabla \boldsymbol{u}^\dagger + (\nabla \boldsymbol{u}^\dagger)^\top\right) - p^\dagger \mathrm{I}_3$ the Cauchy stress tensor, so that the force exerted by the fluid on the swimmer is given by:

$$\mathbf{F}^\dagger(t) = \int_{\partial\mathcal{S}^\dagger(t)} \sigma^\dagger \mathbf{n}\,\mathrm{d}\Gamma\,, \tag{2.9}$$

where $\mathbf{n}$ is the normal directed inwards the domain $\mathcal{S}^\dagger(t)$, see Figure 3.

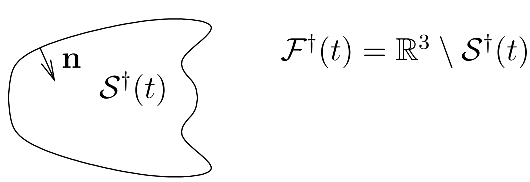


Figure 3: Swimmer into the fluid domain.

2.4 Coupled problem

Without any additional external forces, the motion of the swimmer is given by the Newton law,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{S}^\dagger(t)} v(t,x)\,\rho(t,x)\mathrm{d}x = \mathbf{F}^\dagger(t)\,. \tag{2.10}$$

where for every $x \in \mathcal{S}^\dagger(t)$, we have set:

$$v(t,x) = \dot{h}(t)\mathbf{e}_1 + \dot{\alpha}_1(t) D_1 \left(X(t,\cdot)^{-1}(x - h(t)) \right) \\ + \dot{\alpha}_2(t) D_2 \left(X(t,\cdot)^{-1}(x - h(t)) \right)$$

and where $\mathbf{F}^\dagger(t)$ is given by (2.9). Under the assumption (2.5), the relation (2.10) becomes:

$$m_0 \ddot{h}(t) = \mathbf{F}^\dagger(t) \cdot \mathbf{e}_1 \,, \tag{2.11}$$

with m_0 defined by (2.4).

2.5 Micro-swimmer approximation

For a micro-swimmer such as a protozoa in water, we have $\frac{\mu}{\rho_f} \simeq 10^{-2}\mathrm{cm}^2\mathrm{s}^{-1}$, its length L is of order 10^{-2}cm, its speed U is of order $10^{-1}\mathrm{cm\,s}^{-1}$ and its characteristic time T is of order 10^{-1}s (see [9]). We introduce the Reynolds number $Re = \frac{\rho_f U L}{\mu}$, the time parameter $\tau = \frac{TU}{L}$ and the mass density contrast $C = \frac{1}{\rho_f}$ and using a rescaling of the variables, we obtain the following dimensionless system (see [17, § 5.3] or [9] for technical computations):

- The dimensionless incompressible Navier-Stokes equations:

$$\begin{aligned} Re\left(\tau^{-1}\partial_t \boldsymbol{u}^\dagger + (\boldsymbol{u}^\dagger \cdot \nabla)\boldsymbol{u}^\dagger\right) - \Delta \boldsymbol{u}^\dagger + \nabla p^\dagger &= 0 && \text{in } \mathcal{F}^\dagger(t)\,, \\ \operatorname{div} \boldsymbol{u}^\dagger &= 0 && \text{in } \mathcal{F}^\dagger(t)\,, \end{aligned}$$

  with

$$\lim_{|x|\to\infty} \boldsymbol{u}^\dagger(t,x) = 0\,;$$

- The dimensionless boundary condition:

$$\tau \boldsymbol{u}^\dagger(t,x) = \dot{h}(t)\mathbf{e}_1 + \dot{\alpha}_1(t) D_1 \left(X(t,\cdot)^{-1}(x - h(t)) \right) \\ + \dot{\alpha}_2(t) D_2 \left(X(t,\cdot)^{-1}(x - h(t)) \right)\,,$$

  for every $x \in \partial\mathcal{S}(t)^\dagger$;

- The dimensionless Newton law:

$$\frac{C Re}{\tau^2}\ddot{h}(t) = \mathbf{F}^\dagger(t) \cdot \mathbf{e}_1\,.$$

For the characteristic values of a protozoa, we have $Re \simeq 10^{-1}$, $\tau \simeq 1$ and $C \simeq 1$. Thus taking the limit $Re \to 0$, we formally obtain

- The Stokes equations:

$$-\Delta \boldsymbol{u}^\dagger + \nabla p^\dagger = 0 \qquad \text{in } \mathcal{F}^\dagger(t)\,, \tag{2.12a}$$

$$\operatorname{div} \boldsymbol{u}^\dagger = 0 \qquad \text{in } \mathcal{F}^\dagger(t)\,, \tag{2.12b}$$

  with

$$\lim_{|x|\to\infty} \boldsymbol{u}^\dagger(t,x) = 0\,; \tag{2.12c}$$

- The boundary condition:

$$\boldsymbol{u}^\dagger(t,x) = \dot{h}(t)\mathbf{e}_1 + \dot{\alpha}_1(t)D_1\left(X(t,\cdot)^{-1}\left(x-h(t)\right)\right) \\ + \dot{\alpha}_2(t)D_2\left(X(t,\cdot)^{-1}\left(x-h(t)\right)\right) \quad (2.13)$$

for every $x \in \partial\mathcal{S}^\dagger(t)$;

- Quasi-static Newton law:

$$0 = \mathbf{F}^\dagger(t)\cdot\mathbf{e}_1\,, \quad (2.14)$$

where $\mathbf{F}^\dagger(t) = \int_{\partial\mathcal{S}^\dagger(t)} \sigma(\boldsymbol{u}^\dagger,p^\dagger)\mathbf{n}\,\mathrm{d}\Gamma$ with $\sigma(\boldsymbol{u}^\dagger,p^\dagger) = \left(\nabla\boldsymbol{u}^\dagger + (\nabla\boldsymbol{u}^\dagger)^\top\right) - p^\dagger\mathrm{I}_3$ is the Cauchy stress tensor.

We define the velocity $\boldsymbol{u}$ and the pressure p with the relations $\boldsymbol{u}^\dagger(x) = \boldsymbol{u}(x-h(t)\mathbf{e}_1)$ and $p(x-h(t)\mathbf{e}_1) = p^\dagger(x)$ for every $x\in\mathcal{F}^\dagger(t)$. The system (2.12)-(2.14) becomes

- The Stokes equation:

$$-\Delta\boldsymbol{u} + \nabla p = 0 \qquad \text{in } \mathcal{F}(t)\,, \quad (2.15a)$$

$$\operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } \mathcal{F}(t)\,, \quad (2.15b)$$

where $\mathcal{F}(t) = \mathbb{R}^3\setminus\mathcal{S}(t)$ and with

$$\lim_{|x|\to\infty}\boldsymbol{u}(t,x) = 0\,; \quad (2.15c)$$

- The boundary condition:

$$\boldsymbol{u}(t,x) = \dot{h}(t)\mathbf{e}_1 + \left(\dot{\alpha}_1(t)D_1\left(X(t,\cdot)^{-1}(x)\right)\right. \\ \left. + \dot{\alpha}_2(t)D_2\left(X(t,\cdot)^{-1}(x)\right)\right)\mathbf{e}_r\left(X(t,\cdot)^{-1}(x)\right)\,; \quad (2.16)$$

- Quasi-static Newton law:

$$0 = \mathbf{F}(t)\cdot\mathbf{e}_1\,, \quad (2.17)$$

where $\mathbf{F}(t) = \int_{\partial\mathcal{S}(t)}\sigma(\boldsymbol{u},p)\mathbf{n}\,\mathrm{d}\Gamma$ with $\sigma(\boldsymbol{u},p) = \left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top\right) - p\mathrm{I}_3$.

2.6 Control problem

Let us first notice that in the full system (2.15)–(2.17) the time does not appear directly but only through the parameter $\alpha(t)$. Consequently, we define $\mathcal{S}(\alpha)$ as the image of S_0 by the map $X(\alpha) : x\in\mathbb{R}^3 \mapsto x + \alpha_1 D_1(x) + \alpha_2 D_2(x) \in \mathbb{R}^3$, for $\alpha\in\mathbb{R}^2$ such that:

$$\alpha_1\delta_1(\theta) + \alpha_2\delta_2(\theta) > -1 \qquad (\theta\in[0,\pi])\,, \quad (2.18)$$

which is exactly the condition (2.3) with D_1 and D_2 defined by (2.2). For convenience, we also define the corresponding fluid domain $\mathcal{F}(\alpha) = \mathbb{R}^3\setminus\mathcal{S}(\alpha)$.

For every $\alpha \in \mathbb{R}^2$ satisfying (2.18), we define $(\boldsymbol{u}_0^\alpha, p_0^\alpha)$ the solution of:

$$-\Delta \boldsymbol{u}_0^\alpha + \nabla p_0^\alpha = 0 \qquad \text{in } \mathcal{F}(\alpha)\,, \tag{2.19a}$$
$$\operatorname{div} \boldsymbol{u}_0^\alpha = 0 \qquad \text{in } \mathcal{F}(\alpha)\,, \tag{2.19b}$$
$$\lim_{|x|\to\infty} \boldsymbol{u}_0^\alpha(x) = 0\,, \tag{2.19c}$$
$$\boldsymbol{u}_0^\alpha(x) = \mathbf{e}_1, \quad \text{for all } x \in \partial\mathcal{S}(\alpha) \tag{2.19d}$$

and for $i \in \{1,2\}$, $(\boldsymbol{u}_i^\alpha, p_i^\alpha)$ is the solution of:

$$-\Delta \boldsymbol{u}_i^\alpha + \nabla p_i^\alpha = 0 \qquad \text{in } \mathcal{F}(\alpha)\,, \tag{2.20a}$$
$$\operatorname{div} \boldsymbol{u}_i^\alpha = 0 \qquad \text{in } \mathcal{F}(\alpha)\,, \tag{2.20b}$$
$$\lim_{|x|\to\infty} \boldsymbol{u}_i^\alpha(x) = 0\,, \tag{2.20c}$$
$$\boldsymbol{u}_i^\alpha(x) = D_i\left(X(\alpha)^{-1}(x)\right), \quad \text{for all } x \in \partial\mathcal{S}(\alpha) \tag{2.20d}$$

Then the solution $(\boldsymbol{u}(t,\cdot), p(t,\cdot))$ of (2.15)–(2.16) can be decomposed as

$$\boldsymbol{u}(t,\cdot) = \dot{h}(t)\boldsymbol{u}_0^{\alpha(t)} + \dot{\alpha}_1(t)\boldsymbol{u}_1^{\alpha(t)} + \dot{\alpha}_2(t)\boldsymbol{u}_2^{\alpha(t)}\,,$$
$$p(t,\cdot) = \dot{h}(t)p_0^{\alpha(t)} + \dot{\alpha}_1(t)p_1^{\alpha(t)} + \dot{\alpha}_2(t)p_2^{\alpha(t)}\,.$$

By linearity of the Cauchy stress tensor with respect to $\boldsymbol{u}$ and p, the relation (2.17) becomes:

$$\dot{h}(t)\mathbf{F}_0(\alpha(t)) \cdot \mathbf{e}_1 = -\left(\dot{\alpha}_1(t)\mathbf{F}_1(\alpha(t)) + \dot{\alpha}_2(t)\mathbf{F}_2(\alpha(t))\right) \cdot \mathbf{e}_1\,, \tag{2.21}$$

where for every $i \in \{0,1,2\}$ and every $\alpha \in \mathbb{R}^2$ satisfying (2.18), we have set:

$$\mathbf{F}_i(\alpha) = \int_{\partial\mathcal{S}(\alpha)} \sigma(\boldsymbol{u}_i^\alpha, p_i^\alpha)\mathbf{n}\,\mathrm{d}\Gamma \tag{2.22}$$

with $\sigma(\boldsymbol{u}_i^\alpha, p_i^\alpha) = \left(\nabla \boldsymbol{u}_i^\alpha + (\nabla \boldsymbol{u}_i^\alpha)^\top\right) - p_i^\alpha \mathrm{I}_3$.

Thus the control problem can be recast as:

$$\dot{h}\mathbf{F}_0(\alpha) \cdot \mathbf{e}_1 = -\left(\lambda_1 \mathbf{F}_1(\alpha) + \lambda_2 \mathbf{F}_2(\alpha)\right) \cdot \mathbf{e}_1\,, \tag{2.23a}$$
$$\dot{\alpha} = \lambda\,, \tag{2.23b}$$

where $\lambda = (\lambda_1,\, \lambda_2)^\top \in \mathbb{R}^2$ represents the control of the system whereas $(h,\, \alpha_1,\, \alpha_2)^\top \in \mathbb{R}^3$ is the system's state. Even if α represents the physical control of the swimming system, it is more convenience for analysis to consider $\lambda = \dot{\alpha}$ as the control variable since this will allow to control both the shape of the swimmer and its position.

Let us also set the initial conditions for the system (2.23):

$$h(0) = 0 \quad \text{and} \quad \alpha(0) = 0\,, \tag{2.24}$$

and the target position to be reached in a time $T > 0$:

$$h(T) = h^f \neq 0 \quad \text{and} \quad \alpha(T) = 0\,. \tag{2.25}$$

These initial and final conditions mean that the micro-swimmer is the unit sphere located at the origin at initial time and a unit sphere located in $h^f\mathbf{e}_1$ at final time T.

We point out that a state constraint on α is needed to be able to well define $\mathbf{F}_i(\alpha(t))$ for every time $t \geqslant 0$. This constraint is given by (2.3). But since δ_1 and δ_2 are assumed to be C^1-functions on $[0,\pi]$, we assume the following stronger condition:

$$|\alpha| := \sqrt{\alpha_1^2 + \alpha_2^2} \leqslant \varsigma\,, \tag{2.26}$$

with $\varsigma > 0$ is a given small enough parameter so that if α satisfies (2.26) then the constraint (2.3) is also satisfied

3 Stokes system

In this section, we give some well-posedness and regularity properties on exterior Stokes problem which will be useful for controllability purpose. We first define the weighted Sobolev space $W_0^1(\mathcal{F}(\alpha))$.

Definition 3.1. *Let $\mathcal{S} \subset \mathbb{R}^3$ a compact set with Lipschitz continuous boundary and $\mathcal{F} = \mathbb{R}^3 \setminus \mathcal{S}$. We define the space*

$$W_0^1(\mathcal{F}) = \left\{\varphi \in L^2_{loc}(\mathcal{F})\,,\ \nabla\varphi \in L^2(\mathcal{F})^3\,,\ \sqrt{1+|x|^2}\varphi \in L^2(\mathcal{F})\right\}\,, \tag{3.1}$$

endowed with the norm

$$\|\varphi\|_{W_0^1(\mathcal{F})} = \left\|\sqrt{1+|x|^2}\varphi\right\|_{L^2(\mathcal{F})} + \|\nabla\varphi\|_{L^2(\mathcal{F})^3} \qquad \varphi \in W_0^1(\mathcal{F})\,.$$

The following result is borrowed from [11].

Theorem 3.1. *Let $\mathcal{S} \subset \mathbb{R}^3$ a compact set with Lipschitz continuous boundary and $\mathcal{F} = \mathbb{R}^3 \setminus \mathcal{S}$. For every $\boldsymbol{v} \in H^{\frac{1}{2}}(\partial\mathcal{S})^3$, there exists a unique weak solution $(\boldsymbol{u}, p) \in W_0^1(\mathcal{F})^3 \times L^2(\mathcal{F})$ of the exterior Stokes problem:*

$$-\Delta\boldsymbol{u} + \nabla p = 0 \qquad \textit{in } \mathcal{F}\,, \tag{3.2a}$$

$$\operatorname{div}\boldsymbol{u} = 0 \qquad \textit{in } \mathcal{F}\,, \tag{3.2b}$$

$$\boldsymbol{u} = \boldsymbol{v} \qquad \textit{on } \partial\mathcal{S}\,. \tag{3.2c}$$

In addition, there exists a constant $c(\mathcal{F}) > 0$ such that:

$$\|\boldsymbol{u}\|_{W_0^1(\mathcal{F})^3} + \|p\|_{L^2(\mathcal{F})} \leqslant c(\mathcal{F})\|\boldsymbol{v}\|_{H^{\frac{1}{2}}(\partial\mathcal{S})^3}\,.$$

The limit $\lim\limits_{|x|\to\infty} \boldsymbol{u}(x) = 0$ has to be understood in a weak sense, $\sqrt{1+|x|^2}\boldsymbol{u}(x) \in L^2(\mathcal{F})^3$. In addition, for every $\boldsymbol{u} \in W_0^1(\mathcal{F})$, $\boldsymbol{u}|_{\partial\mathcal{F}} \in H^{\frac{1}{2}}(\mathcal{F})$. Consequently, the expression $\displaystyle\int_{\partial\mathcal{F}} \sigma(\boldsymbol{u}, p)\mathbf{n}\,\mathrm{d}\Gamma$ is seen as a duality product (since $\sigma(\boldsymbol{u}, p)\mathbf{n} \in H^{\frac{-1}{2}}(\partial\mathcal{F})^3$ and $1 \in H^{\frac{1}{2}}(\partial\mathcal{F})$) and by Green formula (to gather with $\operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{u}_0$), we also have:

$$\mathbf{e}_1 \cdot \int_{\partial\mathcal{F}} \sigma(\boldsymbol{u}, p)\mathbf{n}\,\mathrm{d}\Gamma = 2\int_{\mathcal{F}} \mathrm{D}(\boldsymbol{u}) : \mathrm{D}(\boldsymbol{u}_0)\,\mathrm{d}x\,, \tag{3.3}$$

where $(\boldsymbol{u}, p) \in W_0^1(\mathcal{F})^3 \times L^2(\mathcal{F})$ solves (3.2) with boundary condition $\boldsymbol{v} \in H^{\frac{1}{2}}(\partial\mathcal{F})$ and $(\boldsymbol{u}_0, p_0) \in W_0^1(\mathcal{F})^3 \times L^2(\mathcal{F})$ solves (3.2) with boundary condition $\boldsymbol{u}_0 = \mathbf{e}_1 \in H^{\frac{1}{2}}(\partial\mathcal{F})$ and where $\mathrm{D}(\boldsymbol{u})$ is the symmetric strain tensor,

$$\mathrm{D}(\boldsymbol{u}) = \tfrac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\perp}\right).$$

In particular, we have:

$$\mathbf{e}_1 \cdot \int_{\partial\mathcal{F}} \sigma(\boldsymbol{u}_0, p_0)\mathbf{n}\,\mathrm{d}\Gamma = 2\int_{\mathcal{F}} \mathrm{D}(\boldsymbol{u}_0) : \mathrm{D}(\boldsymbol{u}_0)\,\mathrm{d}x > 0\,, \tag{3.4}$$

As a consequence of (3.4) and [18, Lemma 1] or [20, Theorem 2.6], we have the following:

Proposition 3.1. *For $\varsigma > 0$ small enough, the mapping:*

$$\alpha \in \mathbb{R}^2 \mapsto \frac{\mathbf{F}_i(\alpha)\cdot \mathbf{e}_1}{\mathbf{F}_0(\alpha)\cdot \mathbf{e}_1} \in \mathbb{R}^3 \qquad (i \in \{1,2\})\,,$$

with $\mathbf{F}_i$ defined by (2.22), is of class C^∞ on the ball $B_0(\varsigma) \subset \mathbb{R}^2$ centered at 0, with radius ς.

Remark 3.1. *In fact, it is proved in [18, Lemma 1] that this mapping is analytical but in this note we only need the C^∞ regularity.*

4 Controllability and time optimal controllability

Let $\varsigma > 0$ small enough so that the result of Proposition 3.1 holds. For every $\alpha \in B_0(\varsigma)$, we define $V(\alpha) = \dfrac{-1}{\mathbf{F}_0(\alpha)\cdot \mathbf{e}_1}\begin{pmatrix}\mathbf{F}_1(\alpha)\cdot \mathbf{e}_1\\ \mathbf{F}_2(\alpha)\cdot \mathbf{e}_1\end{pmatrix}$. The control system (2.23) can be written as:

$$\dot{h} = V(\alpha)\cdot\lambda\,, \tag{4.1a}$$

$$\dot{\alpha} = \lambda\,, \tag{4.1b}$$

with the initial condition (2.24), the final target (2.25) and the state constraint (2.26). Due to Proposition 3.1, $V \in C^\infty(B_0(\varsigma), \mathbb{R}^2)$ for $\varsigma > 0$ small enough. The following controllability result holds.

Proposition 4.1. *Let $\varsigma > 0$ and $V \in C^\infty(B_0(\varsigma), \mathbb{R}^2)$ be given. Assume that $\nabla V(0) \neq (\nabla V(0))^\top$. Then, for every $h^f \in \mathbb{R}^*$ and every $T > 0$, there exists $\lambda \in C^0([0,T], \mathbb{R}^2)$ such that the solution of (4.1) with the initial condition (2.24) satisfies the final condition (2.25) together with the state constraint (2.26) on α.*

Proof. Let us write $f_1(h,\alpha) = \begin{pmatrix}V_1(\alpha)\\ 1\\ 0\end{pmatrix}$ and $f_2(h,\alpha) = \begin{pmatrix}V_2(\alpha)\\ 0\\ 1\end{pmatrix}$ for $V(\alpha) = (V_1(\alpha), V_2(\alpha))^\top$. The system (4.1) becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}h\\ \alpha\end{pmatrix} = \lambda_1 f_1(h,\alpha) + \lambda_2 f_2(h,\alpha)\,.$$

The Lie bracket of f_1 and f_2 at the point 0 is given by:

$$[f_1, f_2]_0 = \partial_{\alpha_1} V_2(0) - \partial_{\alpha_2} V_1(0)\,,$$

which does not vanishes if $\nabla V(0)$ is not a symmetric matrix. Thus, under this assumption, the Lie algebra generated by $\{f_1, f_2\}$ and evaluated at the point 0 is of dimension 3. In addition, this lie algebra is independent of h and the fist order Lie bracket is continuous with respect to α. Therefore, there exists $\varepsilon > 0$ such that for every $(h, \alpha) \in \mathbb{R} \times B_0(\varepsilon)$, the Lie algebra generated by $\{f_1, f_2\}$ evaluated at the point (h, α) is of dimension 3. The result follows from Chow's Theorem (see for instance [27, chap. 5, Proposition 5.14], [1] or [14]). $\square$

Remark 4.1. *From [20], there exists δ_1, $\delta_2 \in C^1([0,\pi],\mathbb{R})$ such that $\nabla V(0)$ is not symmetric. Moreover, using the arguments of [18], this situation is generic.*

This result combined with the Filipov Theorem (see for instance [1, 8, 13]) leads to the following optimal control result:

Proposition 4.2. *Let $\varsigma > 0$ and $V \in C^\infty(B_0(\varsigma),\mathbb{R}^2)$ such that $\nabla V(0) \neq (\nabla V(0))^\top$. Then, the set of times T such that there exists $\lambda \in BV(0,T)^2$ with $|\lambda(t)| \leqslant 1$ for almost every time $t \in [0,T]$ and such that the solution $(h,\, \alpha)$ of (4.1) with the initial condition (2.24) satisfies the final condition(2.25) together with the state constraint (2.26) on α, admits a minimum value $T^\star = T^\star(h^f, \varsigma)$.*

Remark 4.2. *The constraint $|\lambda(t)| \leqslant 1$ on the deformation velocity λ is necessary to make the time minimal control problem relevant. Without any constraint on the control λ, the minimal time tends to 0 and the corresponding time optimal control does not make sense.*

5 Approximations for small deformations

For a small parameter $\varsigma > 0$, we have $V(\alpha) = V(0) + \nabla V(0)\alpha + o(\varsigma)$ for every $\alpha \in B_0(\varsigma)$. Instead of considering the time optimal controllability of the system (4.1), we first consider the approximated linear system:

$$\dot{h} = (V(0) + \nabla V(0)\alpha)\cdot\lambda\,, \tag{5.1a}$$
$$\dot{\alpha} = \lambda\,, \tag{5.1b}$$

with the initial condition (2.24), the final target (2.25) and with the state constraint (2.26) on α. We assume that the Jacobian matrix of V at $\alpha = 0$ is not symmetric that is $\nabla V(0) \neq (\nabla V(0))^\top$. Then, applying Proposition 4.2 we deduce that there exists a minimal time $T^\star$ such that the solution (h, α, λ) of the control problem (5.1) with the initial condition (2.24) satisfies $h(T^\star) = h^f$, $\alpha(T^\star) = 0$ with the state constraint (2.26) on α. In addition, according to [19, Proposition 7], this optimal time $T^\star$ and time optimal controls can be computed.

Proposition 5.1. *Let $h^f \neq 0$, $\varsigma > 0$ and $V \in C^\infty(B_0(\varsigma),\mathbb{R}^2)$ with $\nabla V(0) \neq (\nabla V(0))^\top$. Let $\gamma \neq 0$ such that $\gamma J = \frac{1}{2}\left(\nabla V(0) - (\nabla V(0))^\top\right)$ with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and*

$$d^\star = \sqrt{\frac{2|h^f|}{\pi|\gamma|}} \qquad \textit{and} \qquad \tau = \frac{\pi\varsigma}{2}\,.$$

Then, the minimal time $T^\star$ of the control problem (5.1),(2.24),(2.25),(2.26) is given by:

$$T^\star = \begin{cases} \pi d^\star & \textit{if } \varsigma \geqslant d^\star\,, \\ \dfrac{\pi (d^\star)^2}{2\varsigma} + \tau & \textit{otherwise.} \end{cases}$$

Moreover, the time optimal control $\lambda^\star$ is continuous and given by:

- *If* $\varsigma \geqslant d^\star$,

$$\lambda^\star(t) = R\left(\operatorname{sign}(h^f)\frac{2\pi\, t}{T^\star}\right)\lambda_0 \qquad (t \in [0, T^\star])\,;$$

- *if* $\varsigma < d^\star$,

$$\lambda^\star(t) = \begin{cases} R\left(\dfrac{\operatorname{sign}(h^f)}{\varsigma}\,2t\right)\lambda_0 & \textit{if } t \in [0,\tau)\,, \\ -R\left(\dfrac{\operatorname{sign}(h^f)}{\varsigma}\,(t-\tau)\right)\lambda_0 & \textit{if } t \in [\tau, T^\star - \tau]\,, \\ -R\left(\dfrac{\operatorname{sign}(h^f)}{\varsigma}\,(2t - T^\star)\right)\lambda_0 & \textit{if } t \in (T^\star - \tau, T^\star]\,. \end{cases}$$

In the above, $\lambda_0 \in \mathbb{R}^2$ *is any vector such that* $|\lambda_0| = 1$ *and for every* $\theta \in \mathbb{R}$*,* $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ *denotes the rotation matrix of angle* θ*.*

Proof. Since the initial and final conditions on α are the same, that is $\alpha(0) = \alpha(T)$, it can be easily proved that λ is a control on $[0, T]$ for the system (5.1), (2.24)–(2.26) if and only if λ is a control on $[0, T]$ for the following system:

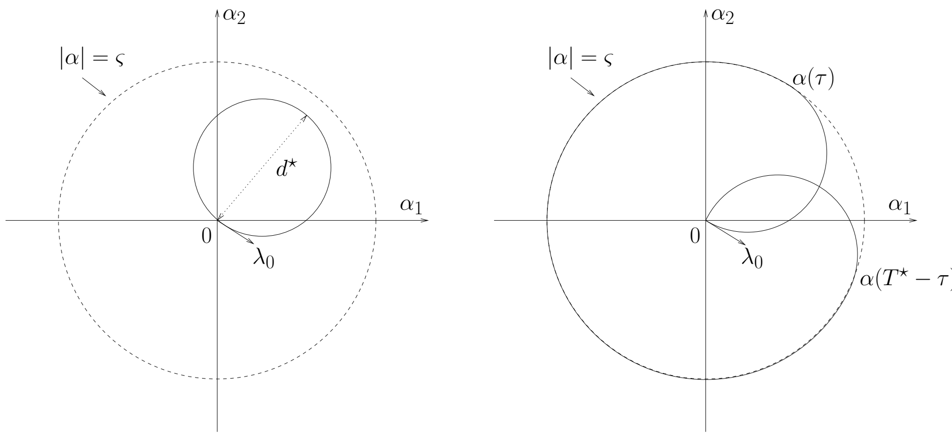
$$\dot{h} = \left(\tfrac{1}{2}\left(\nabla V(0) + (\nabla V(0))^\top\right)\alpha\right)\cdot\lambda\,, \tag{5.2a}$$

$$\dot{\alpha} = \lambda\,, \tag{5.2b}$$

together with (2.24)–(2.26). The result follows from [19, Proposition 7]. □

Remark 5.1. *Let us explain the formulae of the optimal solution given in Proposition 5.1. The optimal deformation* $\alpha^\star(t) = \int_0^t \lambda^\star(s)\,\mathrm{d}s$ *possesses the following characteristics (see Figure 4):*

- *If* $\varsigma \geqslant d^\star$*, the optimal trajectory* $t \mapsto \alpha^\star(t)$ *is a circle of diameter* $d^\star$*, starting from* 0*;*
- *if* $\varsigma < d^\star$*, the optimal trajectory* $t \mapsto \alpha^\star(t)$ *is composed by three arcs of circle. The first arc of circle is an half-circle of diameter* ς*, starting from* 0 *. The second one lies on the circle of diameter* 2ς*, centred on the origin* 0*. Finally, the third arc of circle is an half-circle of diameter* ς*, reaching the origin* 0 *at the final time* $t = T^\star$ *.*



(a) Case $\varsigma \geqslant d^\star$.

(b) Case $\varsigma < d^\star$.

Figure 4: Typical time optimal trajectories of $\alpha = (\alpha_1\ \alpha_2)^\top$.

6 Numerical computation of a time optimal trajectory

Let us consider the deformations D_1 and D_2 given by (2.2) with $\delta_1(\theta) = P_2(\cos\theta)$ and $\delta_2(\theta) = P_3(\cos\theta)$ where P_2 (resp. P_3) is the Legendre polynomial of order 2 (resp. 3). With these two elementary deformations, we obtain from [24],

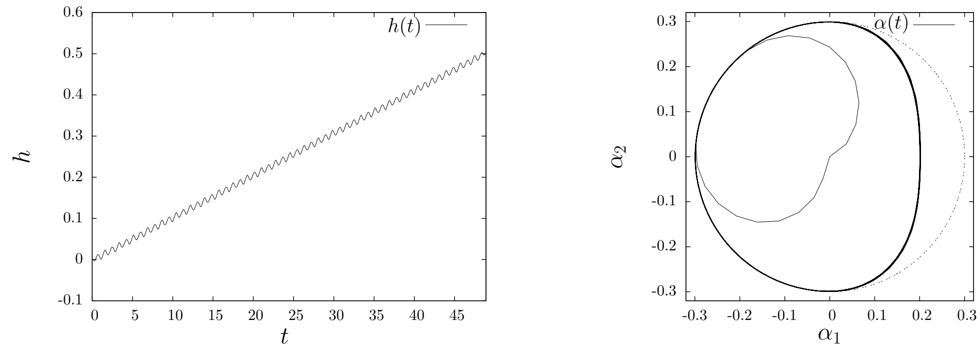
$$V(0) = 0 \qquad \text{and} \qquad \nabla V(0) = \begin{pmatrix} 0 & \frac{6}{35} \\ \frac{4}{15} & 0 \end{pmatrix}.$$

These expressions allow us to compute the explicit form of the time optimal controls for the approximate linearized system (5.1).

In order to numerically compute the fluid forces $\mathbf{F}_i(\alpha) \cdot \mathbf{e}_1$ for $i \in \{0, 1, 2\}$, we use the spherical harmonics expansion of the exterior Stokes solution given in [7] (see also [12] or [24]). Then, we compute the time optimal controls by using a direct method inspired from [27, Chap. 9, Part II, § 1] which is based on the time discretization of the differential equations for h, α and λ. As a result, the time optimal control problem with state constraint is approximated by a nonlinear optimization problem of finite dimension. The initial guess for the discrete nonlinear optimization problem is chosen as the explicit solution of the approximate linear control problem (5.1) given in Proposition 5.1. We expect that the approximate optimal deformation for the linearized problem is close to the optimal solution for the nonlinear problem, so that the direct method will converge quickly.

We apply this computational method with the deformation D_1 and D_2 given above through Legendre polynomials and we choose $\varsigma = 0.3$ and $h^f = \frac{1}{2}$. The optimal trajectories for h and α are depicted on Figure 5. The optimal time is $T^\star \simeq 48.9$.

We numerically observe on Figure 5 that the optimal trajectory for α is mainly periodic. On Figure 6, we also give the optimal trajectory of h during a period in time and finally we plot on Figure 7 the different shapes of the swimmer under the optimal deformation for different instants in the time period .



(a) Optimal trajectory for h.

(b) Optimal trajectory for α.

Figure 5: Optimal trajectories for the system (4.1), (2.24)–(2.26) with deformations $D_1 = rP_2(\cos\theta)\mathbf{e}_r$ and $D_2 = rP_3(\cos\theta)\mathbf{e}_r$, $\varsigma = 0.3$ and $h^f = \frac{1}{2}$.

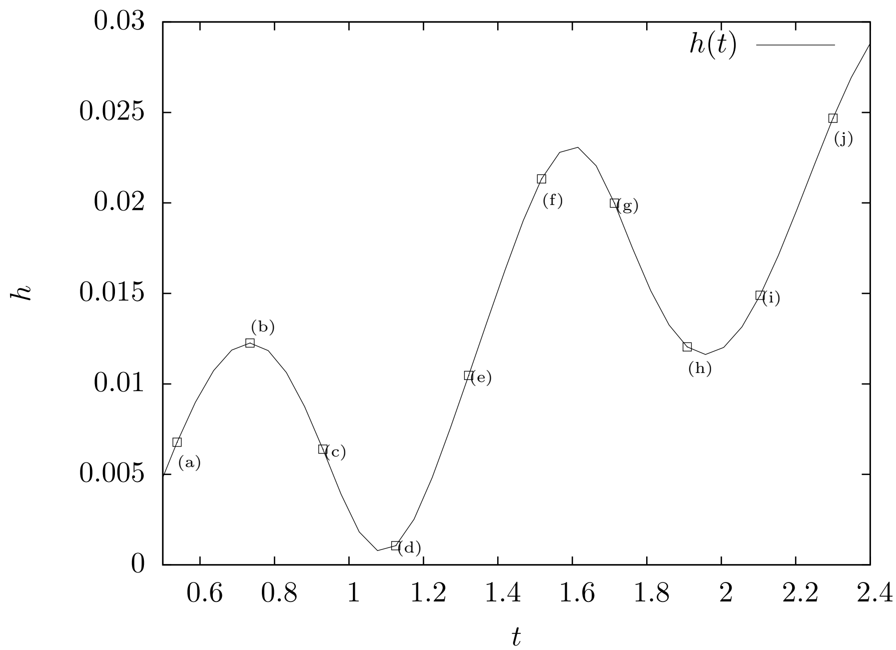
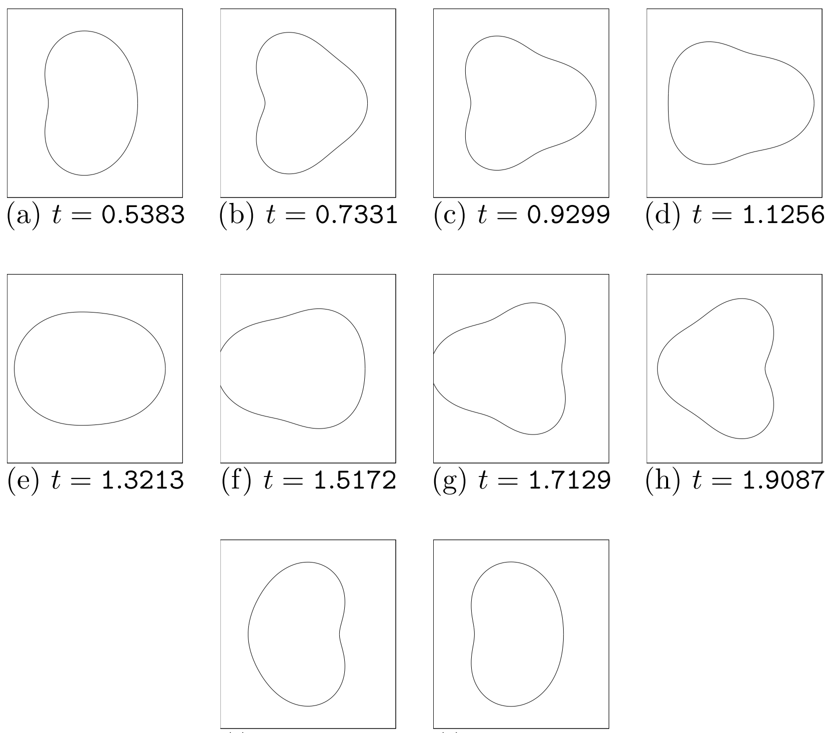


Figure 6: Optimal trajectory of h between $t_0 = \texttt{0.5384}$ and $t_1 = \texttt{2.3002}$ corresponding to one period of α.



(a) $t = 0.5383$ (b) $t = 0.7331$ (c) $t = 0.9299$ (d) $t = 1.1256$

(e) $t = 1.3213$ (f) $t = 1.5172$ (g) $t = 1.7129$ (h) $t = 1.9087$

(i) $t = 2.1035$ (j) $t = 2.3002$

Figure 7: Optimal shapes of the swimmer between $t_0 = 0.5384$ and $t_1 = 2.3002$.

7 Conclusion

In this note we have presented some arguments and results to tackle the numerical resolution of the time optimal swimmer problem when the shapes of the swimmer are small deformations of the unit sphere. Mainly, our strategy is based on the explicit computation of time optimal controls for a state constraint Brockett system which approximates the original nonlinear control problem. This idea could also be applied to the work of F. Alouges, A. DeSimone and L. Heltai [2].

The work presented here is based on two elementary deformations and the swimmer's shapes are close to the unit sphere. However, it is easy to extend this work to the case of a finite number of elementary deformations and where the swimmer's shapes are close to any reference shape.

Finally, we observe numerically that the trajectory for α is mainly periodic. It would be interesting to prove rigorously this property. For instance, one could expect the following result : for $|h^f|$ large enough and for $\alpha \in A$ with A a compact set of $\mathbb{R}^2$, the trajectory of α is composed by one starting curve followed by a periodic curve and ending with a final curve. Such a result would be a real improvement for numerical computations since we would only have to solve three simpler (but coupled) time optimal control problems. In addition, with such a strategy, numerical errors would be reduced.

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